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LETTER TO THE EDITOR

Novel phase-space orbits and quantization

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Abstract
We show that a complex PT-symmetric potential possessing real eigenvalues can also have real and closed phase-space orbits but in a novel way. Further, the phase-space quantization correctly leads to energy-eigenvalues.

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The concept of classical closed phase-space orbits is known to have provided a doorway from classical to quantum and statistical mechanics. This is how the quantum mechanics was developed through various stages. On the other hand, the recent [1, 2] complex PT-symmetric extension of quantum mechanics unlike its conventional counterpart has been discovered almost accidentally. Here the quantum results [1, 6] have come out first and the classical mechanical aspects of such non-Hermitian Hamiltonians remain elusive. Therefore, if it is possible to have the real closed phase-space orbits yet again, we may well go from quantum to classical mechanical aspects of complex PT-symmetric potentials.

This letter intends to provide this missing link by revealing that the complex PT-symmetric potentials can have real closed classical phase-space orbits in a novel way and the phase-space quantization *à la* Wilson and Sommerfeld works again.

Earlier, for two complex PT-symmetric potentials, the WKB quantization employing non-generic transformation of integrals has been found to work [1, 8]. For classical trajectories, classical equations of motion have been solved and portraits of $(x_{\mathrm{real}}, x_{\mathrm{imag}})$ have been plotted and systematized [9, 10]. The Liapunov exponent for a model potential has been found positive to conclude chaos in one dimension [10]. In these investigations, the question of trajectories in the usual sense of phase space, (x, p), has not been addressed. The interesting approach of complex two-dimensional phase space wherein $x \to x_1 + \mathrm{i}p_2$ and $p \to x_2 + \mathrm{i}p_1$ [11] is also not promising in this regard.

First, in the following, we define the new potentials, state and prove certain theorems that will be useful in the following.

PT-symmetric potentials can in general be expressed as

$$V_{\mathrm{PT}}(x) = V_{\mathrm{e}}(x) + \mathrm{i}V_{\mathrm{o}}(x), \tag{1}$$

where $V_e(x)$ and $V_o(x)$ are real and of even and odd parities, respectively. Also $V_e(x)$ is binding potential. Let us also define $\tilde{V}_{PT}(x) = V_{PT}(x + ib)$ this also is PT-symmetric.

Theorem 1. *Both the potentials $\tilde{V}_{PT}(x)$ and $V_{PT}(x)$ possess an identical spectrum.*

Proof. In view of [12], we have the similarity transformation as

$$e^{-bp} V_{PT}(x)\, e^{bp} = V_{PT}(x + ib) = \tilde{V}_{PT}(x). \tag{2}$$

Hence both $V_{PT}(x)$ and $\tilde{V}_{PT}(x)$ will possess an identical spectrum. □

Theorem 2. *The classical turning points of a complex PT-symmetric potential for a real energy, E, essentially occur as $(z, -z^*) : (-a + ib, a + ib)$ or ic, where a, b, c are real numbers.*

(A real equation which is real on real line as per *the fundamental theorem of algebra* has either real or complex-conjugate roots whereas as a complex equation has at least one complex root. Theorem 2 can also be seen as a non-trivial extension of the *fundamental theorem of algebra*).

Proof. The equation, $E = V_{PT}(x)$, determining the classical turning points for a fixed real E can be written as $f_e(x) + i f_o(x) = 0$. Here $f_{e,o}(x)$ are real and of definite parity (even/odd). So if z is a root, we have $f_e(z) + i f_o(z) = 0$. Performing complex conjugation, we get $f_e(z^*) - i f_o(z^*) = 0$ implying that $f_e(-z^*) + i f_o(-z^*) = 0$. This completes the proof. Further, the turning points will be like $(-a + ib, a + ib)$ or purely imaginary like ic. □

As P: $z \to -z$ and T: $z \to z^*$, we note that if z is a turning point, so is $-z^*$. In fact, $-z^*$ is a PT image of z and hence equivalent. Thus $z \to -z^*$ is a periodic orbit [15] of a novel type [7] in a phase space that is PT-symmetrized.

Theorem 3. *For a PT-symmetric potential the action integral*

$$J(E) = \oint p(z)\, dz \tag{3}$$

is real for a real energy. The contour here consists of a straight line path from $z = -a + ib$ to $z = a + ib$ and back.

Proof. The equation of motion for $V_{PT}(z)$ can be written as

$$p(z) = m\frac{dz}{dt} = \pm\sqrt{2m(E - V_{PT}(z))}, \tag{4}$$

let us transform the above equation using $z \to x + ib$, to get

$$p(z) = \pm\sqrt{2m(E - \tilde{V}_{PT}(x))}, \tag{5}$$

$$J(E) = \int_{-a+ib}^{a+ib} +\sqrt{2m[E - V_{PT}(z)]}\, dz + \int_{a+ib}^{-a+ib} -\sqrt{2m[E - V_{PT}(z)]}\, dz \tag{6a}$$

$$J(E) = 2\int_{-a+ib}^{a+ib} \sqrt{2m[E - V_{PT}(z)]}\, dz \tag{6b}$$

$$J(E) = 2\int_{-a}^{a} \sqrt{2m[E - \tilde{V}_{PT}(x)]}\, dx = 2\int_{-a}^{a} [p_r(x) + ip_i(x)]\, dx = 2\int_{-a}^{a} p_r(x)\, dx. \tag{6c}$$

Here

$$p_{\rm r}(x) = \mathrm{Re}\Big(\sqrt{2m[E - \tilde{V}_{\rm PT}(x)]}\Big) \qquad \text{and} \qquad p_{\rm i}(x) = \mathrm{Im}\Big(\sqrt{2m[E - \tilde{V}_{\rm PT}(x)]}\Big). \tag{7}$$

Consequent to the definition of PT-symmetric potentials (1), $p_{\rm r}(x)$ and $p_{\rm i}(x)$ are even and odd functions of x, respectively. This explains the last part of equation (6c). We have phase-space segregated in two parts: $(x, p_{\rm r}(x))$ and $(x, p_{\rm i}(x))$. For a real value of E, the trajectories in the former part will be symmetric about $x = 0$ enclosing a finite area from $x = -a$ to $x = a$. However, in the latter part the trajectories will be essentially antisymmetric about $x = 0$ nullifying the area from $x = -a$ to $x = a$. □

In order to keep the one-dimensional character of $V_{\rm PT}(x)$ intact unlike a previous work [11], at a given energy, E, we absorb the imaginary part of the turning point I$b(E)$ in the potential, $V_{\rm PT}(x)$. Consequently the potential changes to another PT-symmetric potential, $\tilde{V}_{\rm PT}(x)$, that is energy dependent now. However, as per theorem 1, both $V_{\rm PT}(x)$ and $\tilde{V}_{\rm PT}(x + {\rm i}b)$ will have identical spectrum. The remarkable achievement, in going from former to the latter potential, is that the latter one will have real turning points at $x = \pm a$ allowing us to segregate the phase space as concluded above (7)—much in the same way as for the usual real potentials. From theorems (1–3), the following important points emerge.

- It is known that all $V_{\rm PT}(x)$ do not necessarily possess real eigenvalues, they will otherwise possess complex-conjugate pairs of eigenvalues. Interestingly, by looking merely at potential, so far one cannot assess whether there will be a real discrete spectrum. However, now in view of theorems 2, 3 and condition (8), one can ascertain using classical turning points of (1) before going in for the quantization (8) or the exact solution of Schrödinger equation, whether or not there will be a real discrete spectrum. And the necessary condition for this will be the existence of a pair of turning points like $(-a + {\rm i}b, a + {\rm i}b)$ (with $a \neq 0$) for real energies, provided the real part of the potential is a well. For instance, the potential, $V_\nu(x) = -({\rm i}x)^\nu$, since for $0 < \nu \leqslant 1$ the equation, $V_\nu(x) = E$, does not admit any solution moreover real part of the potential is a barrier (not a well). This predicts *a priori* the absence of real discrete spectrum for this potential as observed in the exact calculations in [1]. The real part of $V(x) = ({\rm i}x)^\nu$ is a well but as the equation $(({\rm i}x)^\nu = E, 0 < \nu < 1)$ admits only one root, this potential too does not possess real eigenvalues. More interestingly, the potential $V(x) = -V_0c/(c^2 + x^2) + {\rm i}V_0x/(c^2 + x^2)$, $V_0c > 0$ does not admit real eigenvalues (under Dirichlet boundary condition $\Psi(\pm\infty) = 0$) as one can readily check that it is nothing but $V(x) = -{\rm i}V_0/(x - {\rm i}c)$ entailing only one turning point.
- Eventually, we propose to set $J(E)$ that is nothing but the area enclosed for a fixed orbit in $(x, p_{\rm r}(x))$ in equation (6b) equal to $(n + 1/2)h$ and have a quantization law as

$$\frac{1}{\pi}\int_{-a}^{a} \sqrt{2m(E - V_{\rm PT}(x + {\rm i}b))}\,{\rm d}x = \left(n + \frac{1}{2}\right)\hbar, \tag{8}$$

  such that $V_{\rm PT}(-a + {\rm i}b) = E = V_{\rm PT}(a + {\rm i}b)$ (see theorem 2). In fact for analytic evaluations, one should rather use equation (6b), whereas equation (6c) or (8) is convenient for numerical integrations.
- Now we can express the classical time-period of oscillation associated with the orbits of $(x, p_{\rm r}(x))$ for a fixed energy for a complex PT-symmetric potential. Once again two options are available to us from equation (4):

$$T(E) = \int_{-a+{\rm i}b}^{a+{\rm i}b} \frac{{\rm d}z}{\sqrt{2m[E - V_{\rm PT}(z)]}} = \int_{-a}^{a} \frac{{\rm d}x}{\sqrt{2m[E - V_{\rm PT}(x + {\rm i}b)]}}. \tag{9}$$

We have investigated several PT-symmetric potentials to appreciate the phase-space trajectories and the validity of the quantization law (8). We would like to present here an illustration. In the calculations below, we shall be using $2m = 1 = \hbar$.

Illustration (PT-symmetric Scarf II potential)**.** This complex potential model is immensely useful in PT-symmetric quantum mechanics for its amenability to exact analytic solutions. It was realized quite late that this could be the first exactly solvable model entailing *both real and complex-conjugate pairs* of eigenvalues when the parameter V_2 passes over a critical value of $V_1 + 1/4$ [3], where

$$V_S(x) = -V_1 \operatorname{sech}^2(x) + \mathrm{i}V_2 \operatorname{sech} x \tanh x, \qquad V_1 > 0. \tag{10}$$

This style of writing Scarf II potential has been found more inspiring in later works. The exact eigenvalues of (10) are given as

$$E_n = -\left(n + \tfrac{1}{2} - \tfrac{1}{2}\left[\sqrt{V_1 + V_2 + 1/4} + \sqrt{V_1 - V_2 + 1/4}\right]\right)^2, \tag{11}$$

where $0 \leqslant n \leqslant \frac{1}{2}\left[\sqrt{V_1 + V_2 + 1/4} + \sqrt{V_1 - V_2 + 1/4} - 1/2\right]$. The classical turning points for real and negative energies are given as

$$s_{1,2} = \left(\frac{\mathrm{i}V_2 \pm \sqrt{-V_2^2 - 4E(E + V_1)}}{2E}\right), \tag{12}$$

where $z_{1,2} = \sinh^{-1}(s_{1,2})$. Here $z_{1,2}$ mean $\mp a + \mathrm{i}b$. We now proceed to find the action integral $J(E)$ (6a) for (10). Having used $s = \sinh x$, we have

$$\frac{1}{\pi}\int_{s_1}^{s_2} \frac{\sqrt{Es^2 - \mathrm{i}V_2 s + (E + V_1)}}{1 + s^2}\,\mathrm{d}s = \left(n + \frac{1}{2}\right). \tag{13}$$

Fortunately, this integral is do-able. We see that $\frac{1}{1+s^2} = \frac{1}{2}\left(\frac{1}{1+\mathrm{i}s} + \frac{1}{1-\mathrm{i}s}\right)$, and use the substitutions $1 + \mathrm{i}s = u$ and $1 - \mathrm{i}s = v$ in two integrands to finally find the integrals of the type: $\int \mathrm{d}x\sqrt{ax^2 + bx + c}/x$.

$$E_n = -\left(n + \tfrac{1}{2} - \tfrac{1}{2}\left[\sqrt{V_1 + V_2} + \sqrt{V_1 - V_2}\right]\right)^2, \tag{14}$$

where $0 \leqslant n \leqslant \frac{1}{2}\left[\sqrt{V_1 + V_2} + \sqrt{V_1 - V_2} - 1/2\right]$. Note the typical disappearance of 1/4 in the above WKB-result as compared to the exact quantal result (11). In this illustration, we take $V_1 = 30$, $V_2 = 5$ in arbitrary units, then the potential $V_S(x)$ (10) possesses five real discrete eigenvalues. These WKB (exact) values are $E_0 = -24.58(-24.81)$, $E_1 = -15.66(-15.84)$, $E_2 = -8.75(-8.88)$, $E_3 = -3.83(-3.92)$, $E_4 = -0.91(-0.96)$ as per equations (14) and (11).

We obtain the classical time period for $V_S(x)$ as

$$T(E) = \int_{s_1}^{s_2} \frac{\mathrm{d}s}{\sqrt{Es^2 - \mathrm{i}V_2 s + (E + V_1)}} = \frac{\pi}{\sqrt{-E}}, \tag{15}$$

where $-V_1 \leqslant E < 0$.

We also find that the usage of super-symmetric WKB (SWKB) ansatz [13] using complex supersymmetric superpotential, $W(x)$ [14], yields the exact expressions for eigenvalues of (10) as given in (11), provided we use the PT-symmetric turning points arising from $E = W^2(x)$.

The phase-space orbits are plotted using equation (7) in figures 1 and 2 for three energies which have been chosen as first three eigenvalues. The solid line in these plots denotes the direction from left to right (w.r.t. $x = 0$) when one considers '+' signs in (7). The dashed line denotes the opposite sense of direction (mirror image of the corresponding solid line orbit). The area integral due to solid and dashed parts in $(x, p_{\mathrm{r}}(x))$ will be nonzero (due to symmetry w.r.t $x = 0$) and the area integral for the solid and the dashed curve in $(x, p_{\mathrm{i}}(x))$

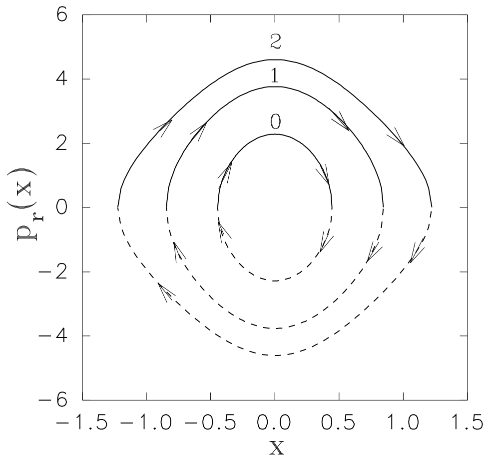

Figure 1. The phase space: $(x, p_{\mathrm{r}}(x))$ for $V_S(x)$ (10), the subscript 'r' stands for 'real' (7). The labels 0, 1, 2 denote the first three eigenvalues (see the text). The solid-line shows a symmetric orbit starting from left to right and the dashed line shows the time-reversed orbit. These two parts make a closed orbit and contribute equally to the area enclosed for a fixed real energy. This area further determines if the given energy is an eigenvalue (see equations (6) and (8)).

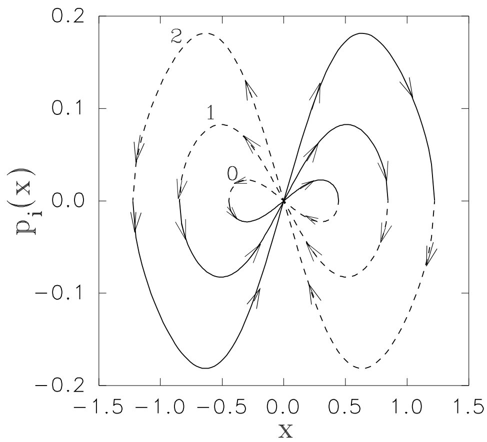

Figure 2. The phase space: $(x, p_{\mathrm{i}}(x))$ for $V_S(x)$, the subscript i stands for 'imaginary' (7). The labels 0, 1, 2 denote the first three energy eigenvalues (see the text). The solid-line shows an antisymmetric orbit starting from left to right and the dashed line shows the time-reversed orbit. The area integral for these two parts vanishes and they make an intersecting orbit not enclosing any area for the fixed real energy.

will essentially vanish (due to antisymmetry w.r.t. $x = 0$). Thus, in the phase space the closed and non-intersecting (intersecting) orbits enclose a finite (zero) area justifying the real discrete spectrum for the PT-symmetric potential.

We have also analysed the Hamiltonians of the type $H = p^2 - (\mathrm{i}x)^\nu$ [1, 4] when $2.0 < \nu < 5.0$ to test the occurrence of closed and smooth phase-space orbits like in figures 1 and 2 leading to correct energy quantization.

The simpler models of complex PT-symmetric potentials such as $V(x) = \frac{x^2}{2} + \mathrm{i}x$ and $V(x) = \cosh x + \mathrm{i}\sinh x$ give rise only to $(x, p_{\mathrm{r}}(x))$ and $p_{\mathrm{i}}(x) = 0$. This can be readily understood as both of them can be rewritten as $V(x + \mathrm{i}c)$ for some real value of c.

The present work eventually leads to a novel segregation of the classical phase space, $(x, p(x))$, in two parts namely, $(x, p_r(x))$ and $(x, p_i(x))$ as displayed in figures 1 and 2, when the potential is complex PT-symmetric. The most serious doubt arising here is whether in doing so there is a loss or discounting of phase-space area as the intersecting and antisymmetric orbits in part $(x, p_i(x))$ do not contribute to the net phase-space area making a way for real eigenvalues. This doubt is dispelled as one finds a fair reproduction of the exact quantal eigenvalues in an approximate way in our illustrations and also in several other model potentials. From this classical analysis, it also turns out that the existence of complex pairs of classical turning points such as $(-a + ib, a + ib)$ (with $a \neq 0$) is the *necessary* condition for a real discrete spectrum of a complex PT-symmetric potential. Equation (8) facilitates the tool to find real eigenvalues for a complex PT-symmetric potential. Having found the real phase-space orbits for the complex PT-symmetric potentials, it would be interesting to find how periodic orbit theory [15] would handle the new potentials in obtaining their real discrete spectrum.

The action integral involved in (8) ought to be carried out in complex plane along a straight line from z to $-z^*$. Singularities occurring on this straight line are specific to the potentials which could also be of exotic nature. Remarkably, while doing the integrations, we find that, even if these points are disregarded, they give rise to a discontinuity in $(x, p_i(x))$ and a non-differentiability in (x, p_r), irrespective of a finite area enclosed in $(x, p_r(x))$. This feature may be usefully interpreted in sorting out spurious eigenvalues. This feature may also warrant the inapplicability or failure of semi-classical method itself for the given potential.

Finally, we conclude by remarking that complex PT-symmetric potentials have been shown here to share even the classical features akin to a real potential. This work opens up a new scope of investigation in both the studies namely, the PT-symmetry and the classical mechanics. We believe that with this, PT-symmetric quantum mechanics passes one of the most stringent tests towards a description that is consistent and compatible with conventional quantum mechanics.

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References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[2] Znojil M (ed) 2003, 2004 *Proc. 1st and 2nd Int. Workshops on 'Pseudo-Hermitian Hamiltonians in Quantum Physics' Czech. J. Phys.* **54**
[3] Ahmed Z 2001 *Phys. Lett.* A **282** 343
[4] Bender C M, Dunne G V and Meisinger P N 1999 *Phys. Lett.* A **252** 272
[5] Ahmed Z 2001 *Phys. Lett.* A **286** 231
[6] Ahmed Z 2004 *Phys. Lett.* A **324** 152
[7] Jain S R unpublished
[8] Bender C M, Berry M V, Meisinger P N, Savage V and Simsek M 2001 *J. Phys. Math. Gen.* **34** L31
[9] Bender C M, Boetcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
[10] Nanayakkara A and Abayaratne C 2002 *Phys. Lett.* A **303** 243
[11] Kaushal R S 2001 *J. Phys. A: Math. Gen.* **34** L209
[12] Ahmed Z 2001 *Phys. Lett.* A **290** 19
[13] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[14] Znojil M and Levai G 2002 *J. Phys. A: Math. Gen.* **35** 8793
[15] Brack M and Bhaduri R K 2003 *Semiclassical Physics* (Oxford: Westview)